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Aerodynamic Coefficients in Generalized Unsteady Thin Airfoil Theory

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Introduction

As shown in Ref. 1, the load induced by an arbitrary (harmonic) disturbance to a thin airfoil can be evaluated explicitly in terms of the "elementary" loads induced by rigid-body heave and pitch. This result is used here to derive simple expressions for the total forces and moments on the airfoil induced by the most important "nonelementary" disturbances: oscillation of a control surface and passage through a sinusoidal gust. Generally subsonic and inviscid conditions are assumed, at least near the airfoil. The results obtained are applicable to a wide variety of outer boundary conditions, nonuniform freestreams, and infinite cascades of identical airfoils. The primary restriction beyond the standard approximations of thin airfoil theory is that the configuration must be translationally invariant in the streamwise (chordwise) direction. In other words, the location of the airfoil in this direction should have no physical significance.

Results and Discussion

Let $w(x)$ be the transverse velocity (upwash) at the airfoil surface, and $P(x)$ be its associated load distribution. All disturbances are assumed to be simple-harmonic in time with reduced frequency k (based on semichord). Thin airfoil theory then provides a linear integral relation between w and P of the form

$$\int_{-1}^{1} dx_1 K(x_1-x)P(x_1)=w(x) \tag{1}$$

which must be solved for the load subject to the Kutta condition

$$P(1)=0 \tag{2}$$

Note that the x coordinate convection here differs from that in Ref. 1.

We shall consider two cases:

1) Rigid body motion of an airfoil-flap combination, consisting of vertical translation of amplitude h, rotation of amplitude α about an axis x_e, rotation of amplitude δ of the control surface alone about its hinge x_h. The upwash for this problem is

$$w(x)=ikh-\alpha[1+ik(x-x_e)]-\delta H(x-x_h)[1+ik(x-x_h)] \tag{3}$$

where H is the unit step function.

2) Sinusoidal gust of amplitude g and wave number γ, for which the upwash is

$$w(x)=ge^{-i\gamma x} \tag{4}$$

The case $\gamma=k$ corresponds to a "simple" gust, i.e., a freestream disturbance which is stationary with respect to the fluid. The load distributions induced by these two types of disturbances will be designated by

$$1)\ P(x)=hP_h(x)+\alpha P_\alpha(x|x_e)+\delta P_\delta(x|x_h) \tag{5}$$

$$2)\ P(x)=gP_g(x|\gamma) \tag{6}$$

In Ref. 1, the heaving and pitching loads, P_h and P_α, were used as basis functions for all other loads. However, these two loads have the unfortunate property of being degenerate at low reduced frequencies ($k \to 0$). Following the suggestion of Kemp,[2] therefore, we introduce the further decomposition

$$P_h(x)=ikP_1(x) \tag{7}$$

$$P_\alpha(x|x_e)=-(1-ikx_e)P_1(x)-ikP_2(x) \tag{8}$$

In summary, then, we shall be dealing with the four pairs (w,P) listed in Table 1.

Each of the loads in Table 1 must individually satisfy the Kutta condition. For these four loads we shall evaluate the following total aerodynamic coefficients:

Lifts: $L_1, L_2, L_\delta(x_h), L_g(\gamma)$, where

$$L=\int_{-1}^{1} dxP(x) \tag{9}$$

Moments: $M_1, M_2, M_\delta(x_h), M_g(\gamma)$, where

$$M=\int_{-1}^{1} dxxP(x) \tag{10}$$

Hinge moments: $N_1(x_h), N_2(x_h), N_\delta(x_h)$, where

$$N=\int_{x_h}^{1} dx(x-x_h)P(x) \tag{11}$$

To do this we must recall the primary result of Ref. 1.

Arbitrary Upwash

Introducing the change in scale and the basis transformation [Eqs. (5) and (6)] into Eq. (12) of Ref. 1, we find that the load induced by an arbitrary upwash w is given by

$$P(x)=\left[P_1(x)\int_{-1}^{1} dx_1 w(-x_1)P_1(x_1) + \frac{d}{dx}\int_{-1}^{1} dx_1 \frac{dw(x_1)}{dx_1}B(x_1,x)\right]\Big/L_1 \tag{12}$$

where B is defined by

$$\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x}\right)B(x_1,x)=P_1(-x_1)P_2(x)-P_2(-x_1)P_1(x)$$

$$B(\pm 1,x)=B(x_1,\pm 1)=0 \tag{13}$$

General algorithms for constructing the load according to Eqs. (12) and (13) are discussed in Ref. 3. Here we confine the discussion to the overall lift and moments. Straightforward

Table 1 Transverse velocities and associated load distributions

$w(x)$	$P(x)$
1	$P_1(x)$
x	$P_2(x)$
$-H(x-x_h)[1+ik(x-x_h)]$	$P_\delta(x\|x_h)$
$e^{-i\gamma x}$	$P_g(x\|\gamma)$

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integration of Eq. (12) provides the relations

$$L = \int_{-1}^{1} \mathrm{d}x W(-x) P_1(x) \tag{14}$$

$$M = \int_{-1}^{1} \mathrm{d}x w(-x) P_2(x) \tag{15}$$

$$N = N_1 L / L_1 + D / L_1 \tag{16}$$

where

$$D = -\int_{x_h}^{1} \mathrm{d}x \int_{-1}^{1} \mathrm{d}x_1 w'(x_1) B(x_1, x) \tag{17}$$

We note that Eqs. (14) and (15) are special cases of more general reciprocity (or "reverse flow") theorems in aerodynamics[4] and could have been obtained directly from the original integral equation [Eq. (1)].

Chebyshev Representations of the Elementary Loads

We shall suppose that the elementary loads $P_{1,2}$ have been approximated by Chebyshev polynomial expansions of degree n:

$$P_j(x) = \frac{1}{\sqrt{1-x^2}} \sum_{k=0}^{n} a_{jk} T_k(x) \quad j = 1,2 \tag{18}$$

with

$$T_j(x) = \cos(j \cos^{-1} x) \tag{19}$$

where the Kutta condition requires that

$$P_j(1) = \sum_{k=0}^{n} a_{jk} = 0 \quad j = 1,2 \tag{20}$$

Standard numerical inversion routines such as collocation or least squares can be used to evaluate the Chebyshev coefficients a_{jk} from the original integral equation (1). Such methods work very well for smooth upwashes, commonly requiring no more than four to six terms for acceptable accuracy. (This is not so, however, of nonsmooth upwashes, as for the oscillating flap.)

Discrete numerical methods, like vortex lattice, of course yield only discrete nodal values, e.g., $P_j(x_k)$, $k = 0, n$ (where without loss of generality we can take $x_n = 1$, $P_j(1) = 0$). If these types of data are available, the Chebyshev weights may be found by curve fitting Eq. (18) to the data. The simplest way to do this is by numerical quadrature of the identity

$$a_{jk} = \frac{2 - \delta_{k0}}{\pi} \int_{-1}^{1} \mathrm{d}x P_j(x) T_k(x) \tag{21}$$

Given the Chebyshev weights of the two elementary loads we can now derive the final expressions for the aerodynamic coefficients.

Gust Response

Substituting the expansions of $P_{1,2}$, of Eq. (18) into the lift and moment expressions, Eqs. (14) and (15) with $w(x) = e^{-i\gamma x}$, we find that

$$L_g(\gamma) = \pi \sum_{j=0}^{n} i^j a_{1j} J_j(\gamma) \tag{22}$$

$$M_g(\gamma) = \pi \sum_{j=0}^{n} i^j a_{2j} J_j(\gamma) \tag{23}$$

where

$$J_j(\gamma) = \frac{i^{-j}}{\pi} \int_{-1}^{1} \mathrm{d}x e^{i\gamma x} \frac{T_j(x)}{\sqrt{1-x^2}} \tag{24}$$

is the Bessel function of order j.

Control Surface

Similarly, for the airfoil-flap combination we find the lifts and moments of the elementary loads:

$$L_1 = \pi a_{10} \tag{25}$$

$$L_2 = \pi a_{20} \tag{26}$$

$$M_1 = (\pi/2) a_{11} = -L_2 \tag{27}$$

$$M_2 = (\pi/2) a_{21} \tag{28}$$

while for the flap mode

$$\begin{pmatrix} L_\delta(x_h) \\ M_\delta(x_h) \end{pmatrix} = \sum_{j=0}^{n} (-1)^{j+1} \begin{pmatrix} a_{1j} \\ a_{2j} \end{pmatrix} [V_j(x_h) + ikW_j(x_h)] \tag{29}$$

where

$$V_j(x_h) = \begin{cases} \cos^{-1} x_h, & j = 0 \\ \dfrac{\sin(j \cos^{-1} x_h)}{j} & j \neq 0 \end{cases} \tag{30}$$

$$W_j(x_h) = \tfrac{1}{2}(V_{j+1} + V_{j-1} - 2x_h V_j) \tag{31}$$

From Eqs. (11) and (18), the hinge moments for the elementary loads are given by

$$N_m(x_h) = \sum_{j=0}^{n} a_{mj} W_j(x_h) \quad (m = 1,2) \tag{32}$$

All of the preceding results follow either directly from the assumed representations of the elementary loads [Eq. (18)] or from the reciprocity rules in conjunction with those representations. The final result, the hinge moment due to flap deflection, involves somewhat more analysis, however. From Eqs. (16) and (17), with the appropriate upwash, we have,

$$N_\delta(x_h) = N_1(x_h) L_\delta(x_h) / L_1 + [D_0(x_h) + ikD_1(x_h)] / L_1 \tag{33}$$

where

$$D_0(x_h) = \int_{x_h}^{1} \mathrm{d}x B(x_h, x) \tag{34}$$

$$D_1(x_h) = \int_{x_h}^{1} \mathrm{d}x \int_{x_h}^{1} \mathrm{d}x_1 B(x_1, x) \tag{35}$$

The D_m integrals can be evaluated explicitly in terms of the Chebyshev coefficients by using the definition of B [Eq. (13)]. Thus, differentiating Eqs. (34) and (35) yields

$$\frac{\mathrm{d}D_0}{\mathrm{d}x_h} = \int_{x_h}^{1} \mathrm{d}x \left(\frac{\partial}{\partial x_h} + \frac{\partial}{\partial x} \right) b(x_h, x) \tag{36}$$

$$\frac{\mathrm{d}D_1}{\mathrm{d}x_h} = -2D_0(x_h) \tag{37}$$

whereby we obtain the alternative formulas

$$D_0 = \int_{x_h}^{1} \mathrm{d}x \int_{x}^{1} \mathrm{d}x_1 [P_2(-x) P_1(x_1) - P_1(-x) P_2(x_1)] \tag{38}$$

$$D_1 = 2\int_{x_h}^{1} \mathrm{d}x D_0(x) \tag{39}$$

Substituting the representations of $P_{1,2}$ from Eq. (18), we find, after some manipulation, that

$$D_m(x_h) = \sum_{i=0}^{n-1} \sum_{j=i+1}^{n} \mathrm{d}_{ij} E_{ij}^{(m)}(x_h) \tag{40}$$

where

$$\mathrm{d}_{ij} = a_{2i}a_{1j} - a_{2j}a_{1i} \tag{41}$$

The E are elementary functions, if rather elaborate ones. They are most compactly expressible in terms of the V_j [Eq. (30)] and the related functions

$$U_j(x_{hj}) = \begin{cases} \tfrac{1}{2}(\cos^{-1}x_h)^2 & (j=0) \\ 1/j^2(1-\cos(j\cos^{-1}x_h) & (j\neq 0) \end{cases} \tag{42}$$

$$\hat{U}_j = jU_j \tag{43}$$

$i+j$ odd

$$E_{ij}^{(0)} = (-1)^i V_i V_j \tag{44}$$

$$E_{ij}^{(1)} = \begin{cases} \dfrac{(-1)^i}{2ij}(\hat{U}_{j-i+1} - \hat{U}_{j-i-1} + \hat{U}_{j+i-1} - \hat{U}_{j+i+1} & (i\neq 0) \\ \dfrac{1}{j}(V_0(V_{j-1} - V_{j+1}) + U_{j+1} - U_{j-1}) & (i=0) \end{cases} \tag{45}$$

$i+j$ even

$$E_{ij}^{(0)} = \begin{cases} (-1)^i \dfrac{i^2-j^2}{2ij}(U_{i+j} - U_{j-i}) & (i\neq 0) \\ 2U_j - V_0V_j & (i=0) \end{cases} \tag{46}$$

$$E_{ij}^{(1)} = \begin{cases} 2(U_1-1)E_{ij}^{(0)} \\ \quad + \dfrac{(-1)^i}{2ij}[(i-j)(\hat{U}_{i+j+1} + \hat{U}_{i+j-1}) \\ \quad + (i+j)(\hat{U}_{j-i+1} + \hat{U}_{j-i-1})] \quad (i\neq 0) \\ 4U_1/j^2 + \dfrac{1}{j^2}[(3j-2)U_{j-1} - (3j+2)U_{j+1}] \\ \quad - \dfrac{1}{j}V_0(V_{j-1} - V_{j+1}) \quad (i=0) \end{cases} \tag{47}$$

Conclusions

Simple universal formulas have been given for the most important aerodynamic coefficients in unsteady thin airfoil theory. The lift and moment induced by a generalized gust are evaluated explicitly in terms of the gust wavelength. Similarly, in the control surface problem, the lift, moment, and hinge moments are given as explicit algebraic functions of hinge location. These results can be used in conjunction with any of the standard numerical inversion routines for the elementary loads (pitch and heave).

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Separation Pressure of a Turbulent Boundary Layer in Transonic Interactions

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Introduction

A SEMIEMPIRICAL method for the calculation of the downstream characteristics of a boundary layer interacting with a shock is called "discontinuity analysis." The basis of this method is the omission of the shear stress term from the x-momentum equation as being negligible compared with the pressure gradient term, and the derivation of equations which give the overall development of some critical characteristics of the boundary layer under the impact of a given pressure jump Δp, the strength of the shock. In this kind of analysis no length scale appears and, apart from the convenience of ignoring the length of the interaction, which is an unknown quantity, this independence from length scales simplifies the equations and provides similarity relations which efficiently describe the physical mechanisms controlling the overall development of the boundary layer.

The most widely known discontinuity analysis is the one of Reshotko and Tucker,[1] while Green[2] has classified the various approaches which are based on this principle and justified their use by presenting comparisons with experimental data.

More recently, the present author has developed a new discontinuity analysis[3] in which the similarity relations giving the development of the thickness quantities δ^* and ϑ are made more precise through inclusion of the thickening of the boundary layer itself. As a result, the agreement with the experimental evidence is better than the methods reviewed by Green. An additional difference between our analysis and the method of Reshotko and Tucker is the possibility in the former approach of including mass entrainment, which in the higher transonic numbers is significant, while the latter approach is based strictly on the zero entrainment principle.

Apart from the downstream characteristics of an interacting turbulent boundary layer,[4,5] our analysis provides a new incipient separation criterion and gives an explanation for the observed behavior of the pressure at the separation point, for increasing upstream Mach number, and for varying initial conditions of the boundary layer. In this Note those latter aspects of the interaction problem will be presented.

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